# Solitons in a Bilocal Field Theory

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ABSTRACT

We obtain a bilocal classical field theory as the large $N$ limit of the chiral Gross–Neveu (or non–abelian Thirring) model. Exact classical solutions that describe topological solitons are obtained. It is shown that their mass spectrum agrees with the large $N$ limit of the spectrum of the chiral Gross–Neveu model.

We will study the large $N$ limit of the two– dimensional fermionic system defined by the Lagrangian

$$L = \bar{q}^i[-i\gamma_\mu\partial^\mu]q_i + \frac{g^2}{2N}\bar{q}^i\gamma_\mu q_j\bar{q}^j\gamma^\mu q_i. \tag{1}$$

Here $i = 1, \cdots N$. This model is a non–abelian version of the Thirring model. It is also known as the chiral Gross–Neveu [1] model since the interaction can be rewritten in the following form using a Fierz identity.

$$L = \bar{q}^i[-i\gamma_\mu\partial^\mu]q_i - \frac{g^2}{2N}[\bar{q}^i q_i\bar{q}^j q_j - \bar{q}^i\gamma_5 q_i\bar{q}^j\gamma_5 q_j]. \tag{2}$$

The precise definition of the model requires a renormalization of the coupling constant. This theory is asymptotically free, the dimensionless coupling constant $g^2$ is replaced by a dimensional constant as the true parameter upon renormalization. This model has been solved exactly [2], [3] by Bethe ansatz methods; i.e., its spectrum and the $S$-matrix are known explicitly. In the limit as $N \to \infty$, it should tend to an exactly integrable classical theory. This classical theory will require a renormalization in order to be well–defined ( the beta–function does not vanish in the large $N$ limit). Still, we should be able to understand the large $N$ limit of the mass spectrum in terms of the classical solutions. Such a direct understanding of the large $N$ limit of this type of theories has not been achieved yet, as pointed out in [4]. Berezin [5] has given a bi–local formulation of the large $N$ limit of our type of model; however, the mass spectrum was not obtained in that reference. What amounts to the linear approximation to our theory has been studied in a recent paper [6].

In this paper we will obtain the classical theory corresponding to the large $N$ limit of this model as well as obtain its mass spectrum, in agreement with the known exact solutions. We will apply the bilocal bosonization method [7], [8], [9] as well as the non–perturbative renormalization methods [10] developed in previous papers to this model. Having established the validity of these methods in this exactly solvable context, these methods can in the future be used in more realistic models such as spherical QCD [8]. In fact, one of the lessons of this paper is how to define classical field theories which require a renormalization. We believe this is the first example of exact soliton solutions in a bilocal field theory. We also discover some new phenomena such as a topological charge with a continuous range of values.

The large $N$ limit of our model can be studied by the technique of summing an appropriate class of Feynman diagrams. As is clear from other examples, [11] this will produce a free field theory of small oscillations around the vacuum, the analogues of

mesons in two dimensional QCD. However, it is clear from those examples that the true large $N$ limit is a highly non–linear classical theory. In particular it can have classical solutions which are topological solitons and hence are large deviations from the vacuum. The bi–local bosonization methods of Refs. [7] can construct the complete large $N$ limit and describe even these topological solitons [12]. ( In 2dqcd they are the baryons.) Indeed we will see that in the case of the chiral Gross–Neveu model, *all* the particles in the spectrum are topological solitons. A surprise is that the topologically conserved quantum number can take continuous values between 0 and $2\pi$. A completely precise mathematical description of such a topological invariant, with a continuous range, requires the theory of projectors on von Neumann algebras and is beyond the scope of this paper. The technical apparatus necessary for this seems to exist already [13].

Let us begin by understanding the symmetries of the above lagrangian. First of all it has a global symmetry under $U(N) \times U(N)$, the first (second) $U(N)$ acting on the left (right) components of $q$. This non–abelian part of this chiral symmetry is unbroken and the particles of theory transform under the completely anti– symmetric tensor representations of $SU(N) \times SU(N)$. ( This is unlike in 2dqcd, where all particles are in the trivial representation of $SU(N)$ due to confinement.) If $r = 1, \cdots, N-1$ is the rank of the tensor, the particles transform under the representation $(r, r)$ of $SU(N) \times SU(N)$ and have masses [3], [2]

$$m_r = m_1 \frac{\sin \frac{\pi r}{N}}{\sin \frac{\pi}{N}}. \tag{3}$$

The quantum number $r$ is also a conserved quantity of the fermionic lagrangian, corresponding to the symmetry group $Z_N$. We might understand this discrete symmetry by considering the $U(1)$ sub–group of the global symmetry corresponding to

$$q_i \to e^{i\alpha} q_i. \tag{4}$$

If $\alpha$ is an integer multiple of $\frac{2\pi}{N}$, this transformation is in fact a particular element in $SU(N) \times SU(N)$. This $Z_N$ subgroup may viewed as measuring the fractional part of fermion number, if $q_i$ is assigned a fermion number of $\frac{1}{N}$. ( This is analogous to the 'triality' of $SU(3)$ representations.) The part of fermion number which is integral can be viewed as a separate conserved quantity, whose $U(1)$ symmetry group has no overlap with $SU(N) \times SU(N)$. We will thus decompose fermion number into two conserved quantities, one taking integer values on each completely occupied one–particle state (which we will call baryon number $B$ in analogy with 2dqcd) and another, $\theta$ taking values $0, 2\pi\frac{1}{N}, 2\pi\frac{2}{N}, \cdots, 2\pi[1-\frac{1}{N}]$ corresponding to $Z_N$.

In the Dirac sea, any one–particle state that is completely filled will contain $N$ fermions and will make no contribution to the conserved charge $\theta$ of this $Z_N$ symmetry. If a state is partially filled, its contribution will be equal to the ratio of the number of occupied states to the number of available states, which is an integer multiple of $\frac{1}{N}$. The conserved quantity $\frac{r}{N}$ is precisely this fractional part of the baryon number. Obviously, the 'fractional part' of the fermion number can add up to give an integer if we sum over many states. Less obvious is that the integral part of the fermion number can add up to a fractional value when summed over an infinite number of states; this has to do with regularizations that are necessary to make such a sum well–defined [14].

There are several possible ways of taking the limit $N \to \infty$. For example, we could take the limit keeping the mass $m_1$ of the lightest particle fixed. Then, the spectrum would consist of two infinite towers of particles of equally spaced masses; one tower corresponding to $r = 1, \cdots, \frac{N}{2}$ and the other to $r = N-1, N-2, \cdots \frac{N}{2}+1$. ( This is for $N$ even; there is a similar expression for $N$ odd.) A more interesting large $N$ limit will be obtained if we take the limit keeping the mass $\mu$ of the heaviest particle fixed. Then the mass differences between particles tends to zero in the large $N$ limit and the spectrum merges into a continuum. (Such limits of the exact solutions have been studied before in a somewhat different context. [15].) Also, in this limit, the variable $\theta$ takes a continuous range of values between 0 and $2\pi$ so that the mass spectrum becomes the continuous set,

$$m_\theta = \mu \sin\frac{\theta}{2}, \quad \text{for} \quad 0 \leq \theta \leq 2\pi. \tag{5}$$

We will try to understand this limit as a classical field theory. We will see that $\theta$ corresponds to a topologically conserved quantity with a continuous range of values between 0 and $2\pi$ in the classical theory.

First we will give a heuristic derivation of the classical theory corresponding to the large $N$ limit of the Thirring model. Then we will give a more rigorous *ab initio* definition of this classical theory and find some static solutions to it. We regard the following discussion of the large $N$ limit only as motivation for the study of this classical theory. A more rigorous justification would be in terms of the quantization of the classical theory, along lines discussed in [7]. $\frac{1}{N}$ will play the role of $\hbar$ in this quantization.

We will study the theory in hamiltonian form, in which the field operators satisfy the equal time anti–commutation relations

$$[q^{\dagger i}_\alpha(x), q_{j\beta}(y)]_+ = \delta_{\alpha\beta}\delta^i_j\delta(x-y). \tag{6}$$

$\alpha, \beta = 1, 2$ are spin indices, which we will find convenient to suppress usually. If we ignore delicate issues of renormalization, the hamiltonian can be expressed in terms of the color singlet bilinear $M(x,y) = -\frac{1}{N} : q_i(x)q^{\dagger i}(y) :$ ( here, $M(x,y)$ is a $2\times 2$ matrix in spin space and : : denotes normal ordering with respect to the free hamiltonian):

$$H = \int \left[ \mathrm{tr}(-i\gamma_5)\left[\frac{\partial M(y,x)}{\partial y}\right]_{x=y} - \frac{g^2}{2}\mathrm{tr}\left[M^2(x,x) - \gamma_5 M(x,x)\gamma_5 M(x,x)\right]\right] dx. \qquad (7)$$

Conventional large $N$ arguments can be used to show that the observables $M(x,y)$ will have quantum fluctuations of order $\frac{1}{N}$, so that their time evolution will be described by classical equations of motion. (One way to see this is to note that their commutators are of order $\frac{1}{N}$ [7]. They form a representation of the infinite dimensional unitary Lie algebra, also called $W_\infty$ algebra in the context of matrix models.) In a theory such as 2dqcd which has color confinement, these would be a complete set of observables. However, the physical states of our theory transform under non–trivial representations of color, so that these are not a complete set of observables.

We can now see that $M(x,y)$ along with the charges of the $SU(N)\times SU(N)$ symmetry are in fact a complete set of observables. It is best to understand this fact in a regularized context in which the position variables are allowed to take only a finite number ( say $K$) of values. The complete set of bilinears $\Phi^{j\beta}_{i\alpha}(x,y) = -i[q^{i\alpha}(x), q^{\dagger j\beta}(y)]$ form the Lie algebra $\underline{U}(2KN)$ under commutation. The Fermionic Fock space carries an irreducible representation of this Lie algebra. This Lie algebra contains $\underline{SU}(N)\times \underline{SU(N)}\times \underline{U}(K)$ as a subalgebra. The generators of this subalgebra are the charges of the global symmetry $\underline{SU}(N)\times \underline{SU}(N)$ and the operators $M(x,y)$ themselves. The key fact is that the representation of $U(2KN)$ on the fermionic Fock space remains irreducible with respect to this subalgebra. ( We omit a detailed proof in the interest of brevity.) Thus any operator that commutes with the charges of the $SU(N)\times SU(N)$ symmetry as well as $M(x,y)$ is a multiple of the identity: these together form a complete set of observables. Thus the dynamics of the theory reduces to a classical dynamics for $M(x,y)$; the generators of the global symmetry never become classical, but they have trivial dynamics, being conserved quantities. Thus we will for the most part concentrate on the variables $M(x,y)$. These arguments can probably be made rigorous in the case were $x, y$ have infinite range, but the techniques required probably involve use of $C^*$ or von Neumann algebras. We will not attempt it in this paper.

Any classical theory is described by a phase space ( manifold of allowed initial conditions), a symplectic structure on this manifold ( or, Poisson brackets of a complete set

of dynamical variables) and a hamiltonian. In some cases, the phase space is described as the set of solutions to some set of constraints satisfied by some dynamical variable, rather than directly in terms of a co-ordinate system.

In our classical theory, the dynamical variables are self- adjoint operators $M$ on the complex Hilbert space $L^2(R, C^2)$. ( This can be thought of the one-particle Hilbert space of a Dirac fermion in 1+1 dimensional space- -time.) $M$ can be described in terms of its kernel,

$$Mu(x) = \int M(x,y)u(y)dy \tag{8}$$

where $M(x,y)$ is a $2 \times 2$ matrix valued function ( in general distribution) of the two variables $x, y$. An equivalent description is in terms of the 'symbol', $\tilde{M}(x,p)$, which is a Fourier transform of the kernel with respect to the relative co–ordinate:

$$\tilde{M}(x,p) = \int M(x+\frac{y}{2}, x-\frac{y}{2})e^{-ipy}dy. \tag{9}$$

Since $M$ is self–adjoint, $\tilde{M}(x,p)$ is real valued. Spatial translation acts as follows on these variables: $M(x,y) \to M(x+a, y+a)$ and $\tilde{M}(x,p) \to \tilde{M}(x+a,p)$. We can regard $\tilde{M}(x,p)$ as an infinite component classical field, labelled by a continuous internal index $p$. Such bilocal field variables have been useful in formulating the large $N_c$ limit of models for QCD.

In spite of the fact that operators on Hilbert spaces make an appearance, we emphasize that ours is still a classical theory, the dynamical variable of which just happens to be an infinite dimensional 'matrix' $M(x,y)$. The Poisson brackets satisfied by these variables are,

$$\begin{aligned}\{M^\beta_\alpha(x,y), M^\delta_\gamma(z,u)\} = {} & \delta^\beta_\gamma\delta(y-z)[M^\delta_\alpha(x,u) + \epsilon^\delta_{0\alpha}(x,u)] \\ & - \delta^\delta_\alpha\delta(u-x)[M^\beta_\gamma(z,y) + \epsilon^\beta_{0\gamma}(z,y)].\end{aligned}$$

Here $\alpha, \beta$ are spin indices which we usually suppress. Also,

$$\epsilon^\beta_{0\alpha}(x,y) = \gamma^\beta_{5\alpha}\int \frac{dp}{2\pi}e^{ip(x-y)} \text{ sgn } (p) \tag{10}$$

is the kernel of an operator $\epsilon_0$ whose square is one. It has the physical meaning of being sign of the massless Dirac operator; the eigenspace of $\epsilon_0$ with eigenvalue 1 (or -1) is the space of positive (or non–positive ) energy one–particle states.

This algebra is the central extension of the unitary Lie algebra, called the Lundberg–Kac–Petersen [16] extension. This algebra has also acquired the name $W_\infty$ algebra in the context of matrix models. The above Poisson brackets are just the commutation

relations of the operators $M(x,y)$ introduced previously, except that a factor of $\frac{i}{N}$ has been removed. This is the appropriate prescription as $\frac{1}{N}$ plays the role of $\hbar$ in our classical limit. The central term proportional to $\epsilon_0$ appears because of the normal ordering of $M$ in the quantum theory.

We will impose some conditions on the asymptotic behaviour of $\tilde{M}(x,p)$ as $|p| \to \infty$. These follow from the asymptotic behaviour of the matrix elements of

$$\frac{1}{N_c}\int : q_i(x+\frac{y}{2})q^{\dagger i}(x-\frac{y}{2}) : e^{-ipy}dy$$

in free fermion theory. Due to asymptotic freedom, this is the same as the behaviour of the interacting theory. However in the classical theory they are to be viewed as postulates.

We will impose

$$\tilde{M}_d(x,p) \sim O(\frac{1}{|p|^2}) \quad \tilde{M}_{od}(x,p) \sim O(\frac{1}{|p|}) \tag{11}$$

where $\tilde{M}_d(x,p)$ (or, $\tilde{M}_{od}(x,p)$) is the diagonal (or, off–diagonal) part of the $2\times 2$ matrix $\tilde{M}(x,p)$ in a basis where $\gamma_5$ is diagonal. It will not be possible to absorb the central term into the definition of $M(x,y)$ without violating these conditions; this is what makes the central extension non–trivial. The above commutation relations then define a topological Lie algebra, the topology being defined by the norm implicit in this asymptotic behaviour:

$$||M|| = \sup_{x,p}[(p^2+1)|\tilde{M}_d(x,p)|] + \sup_{x,p}[(|p|+1)|\tilde{M}_{od}(x,p)|]. \tag{12}$$

The phase space of the theory is a co–adjoint orbit of this Lie algebra ( or rather the corresponding Lie group, which is a dense subgroup of the restricted Unitary group $U_1(H)$ of Segal [17]; we refer to [7] for a more detailed description). In the case of 2dqcd, this orbit was a Grassmannian, which is defined by a quadratic equation satisfied by $M$:

$$[M+\epsilon_0]^2 = 1. \tag{13}$$

The co–adjoint orbits of the restricted unitary group are known; they are infinite dimensional analogues of the familiar flag manifolds. ( This theory is not necessary in the following, except as motivation for some definitions.) We will require that $M$ satisfy the above constraint except for a finite dimensional block. More precisely,that

$$[M+\epsilon_0]^2 - 1 \text{ is finite rank.} \tag{14}$$

For the static solutions we are interested in, this block is in fact one-dimensional. The meaning of this condition is that we allow for fermion states to be completely filled or completely unfilled except for a finite dimensional block of states which may be only partially filled. If in fact $(M+\epsilon_0)^2 = 1$ all the states in the theory would be either completely filled or completely unfilled and therefore singlets under the global $SU(N)$ symmetry. [18] This is too strong a condition in our case. In fact the constraint that $(M+\epsilon_0)^2 - 1$ be finite rank will enforce that

$$\tilde{M}^2(x,p) \sim -\text{ sgn }(p)[\gamma_5, \tilde{M}(x,p)]_+ \tag{15}$$

or that

$$\tilde{M}_d^2(x,p) + \tilde{M}_{od}^2(x,p) \sim -\text{ sgn }(p)[\gamma_5, \tilde{M}_{od}(x,p)]_+. \tag{16}$$

These asymptotic conditions will be useful later.

The hamiltonian of the theory can be obtained by rewriting the regularized fermionic hamiltonian in terms of the bilinears. It is convenient to write it in terms of $\tilde{M}(x,p)$, for then a cut–off in the range of $p$ provides a natural regularization. After some manipulations we can bring the hamiltonian to the form,

$$E_\Lambda(M) = \int dx \big[ \int_{|p|<\Lambda} \frac{dp}{2\pi} \text{tr}[\gamma_5 p \tilde{M}(x,p) + \frac{1}{2} V(x) \tilde{M}(x,p)] - C \big]. \tag{17}$$

Here,

$$V(x) = 2g^2(\Lambda) \int_{|p|<\Lambda} \tilde{M}_{od}(x,p) \frac{dp}{2\pi} \tag{18}$$

and $g^2(\Lambda)$ is a coupling constant which will be picked such that the theory is well–defined ( see below). Also $C$ is picked such that the vacuum solution has zero energy density. The equations of motion are anyway independent of $C$.

It is clear from the asymptotic behaviour of $\tilde{M}_{od}(x,p)$ that , if $g^2(\Lambda) \sim \frac{g_1^2}{\log \Lambda}$, $V(x)$ will be independent of $\Lambda$ asymptotically. In fact, this will turn out to be a self– consistent choice for $g^2(\Lambda)$. If moreover, $g_1^2 = \frac{\pi}{2}$, the quantity

$$\text{tr}[\gamma_5 p \tilde{M}_d(x,p) + \frac{1}{2} V(x) \tilde{M}_{od}(x,p)] \tag{19}$$

vanishes faster than $\frac{1}{|p|}$. ( Each term in the trace goes like $\frac{1}{|p|}$ but the leading contributions cancel.) This means that the energy can be defined by taking the limit $\Lambda \to \infty$,

$$E(M) = \int dx \big[ \int \frac{dp}{2\pi} \text{tr}[\gamma_5 p \tilde{M}(x,p) + \frac{1}{2} V(x) \tilde{M}(x,p)] - C \big]; \tag{20}$$

a regularization is only necessary in the above expression for $V(x)$ in terms of $\tilde{M}(x,p)$.

The equations of motion that follow from this hamiltonian and the above commutations relations are, $\frac{\partial M(x,y;t)}{\partial t} = \{E(M), M\}$. The Poisson bracket relations being those of a unitary Lie algebra, the r.h.s. can be written in terms of the commutator of operators. If we use operator notation, we get,

$$\frac{\partial M(t)}{\partial t} = [H(M), M + \epsilon_0] \tag{21}$$

where the operator $H$ is defined to be the derivative of $E$ with respect to $M$,

$$H(M) = \frac{\partial E}{\partial M}. \tag{22}$$

Explicitly, $H(M)$ is a differential operator,

$$H(M) = -i\gamma_5 \frac{\partial}{\partial x} + V(x) \tag{23}$$

where $V(x)$ is related to $M$ by the equation given earlier. Static solutions must therefore satisfy the nonlinear equation

$$[M + \epsilon_0, H(M)] = 0. \tag{24}$$

Since we are studying the large $N$ limit of a theory which is exactly solvable for every $N$, there must be a way to solve these equations as well: the above classical dynamical system must be integrable. We do not attempt to demonstrate the exact integrability of this system here; instead we will obtain a family of classical static solutions that describe a kind of topological soliton. We will show that the mass spectrum of this soliton agrees with the known large $N$ limit of the mass spectrum of the Thirring model.

A simple solution to the above equations of motion is,

$$\tilde{M}(x,p) + \epsilon_0(p) = \frac{\gamma_5 p + m\gamma_0}{\sqrt{(p^2 + m^2)}} \tag{25}$$

with $V(x) = m\gamma_0$. The relation between $V(x)$ and $M$ is satisfied if

$$2g^2(\Lambda) \int_{|p|<\Lambda} \frac{dp}{2\pi} \frac{1}{\sqrt{(p^2 + m^2)}} = 1. \tag{26}$$

This solution describes the vacuum, being translation invariant. From now on we will assume that $g^2(\Lambda)$ is given by the above equation. We have traded the dimensionless

coupling constant $g$ for the dimensional constant $m$. The constant $C$ is now fixed so that this solution has zero energy:

$$C = 2\int \frac{dp}{2\pi}\Big[p\{\frac{p}{\sqrt{(p^2+m^2)}} - \text{ sgn }(p)\} + \frac{m^2}{2\sqrt{(p^2+m^2)}}\Big] \tag{27}$$

which is a convergent integral.

In fact the vacuum is degenerate; we could have replaced it by a chirally rotated solution,

$$V(x) = e^{-i\gamma_5\frac{\theta}{2}} m\gamma_0 e^{i\gamma_5\frac{\theta}{2}}, \qquad \tilde{M}(x,p) + \epsilon_0(p) = e^{-i\gamma_5\frac{\theta}{2}} \frac{\gamma_5 p + m\gamma_0}{\sqrt{(p^2+m^2)}} e^{i\gamma_5\frac{\theta}{2}} \tag{28}$$

which would have the same energy ( zero). This suggests the possibility of more general solutions which depend on $x$ and tend to two different vacua as $|x| \to \pm\infty$. Such solutions would describe topological solitons of the bi–local theory. Notice that there is a continuous infinity of vacua and thus the solitons are parametrized by an angle $\theta$, measuring the difference between the directions of the vacua at infinity. We do not know of a systematic way to search for such static solutions of the above classical equations for $M$. Instead we will produce a solution by a guess inspired by the theory of solitons of the non–linear Schrodinger equation.

Now, $H$ is the Dirac operator of a massive spin $\frac{1}{2}$ particle coupled to an external scalar field $V(x)$. $H$ has both a point ( discrete) spectrum corresponding to bound states and a continuous spectrum corresponding to scattering states. The discrete spectrum will have eigenvalues in the range $-m < \lambda < m$ and will have eigenfunctions which are square integrable.The continuous spectrum ( 'scattering states') have eigenvalues with $\lambda \leq -m$ or $\lambda \geq m$. There is no normalizable eigenvector corresponding to these; however, it is still meaningful to speak of the projection operator $P_I$ to the subspace with eigenvalues within some interval $I$ of the real axis. This spectral projection operator $P_I$ can be written as

$$P_I = \int_I \frac{d\lambda}{2\pi}\rho(\lambda) \tag{29}$$

where the 'spectral density' $\rho(\lambda)$ is in general some operator– valued distribution. The contribution of a bound state will involve a delta-function in $\lambda$ while that of the continuum will be some continuous function in $\lambda$. In terms of the spectral density, we have a decomposition,

$$H = \int \frac{d\lambda}{2\pi}\lambda\rho(\lambda) \tag{30}$$

which is the generalization of the familiar decomposition of a matrix into eigenvalues and eigenvectors. We can determine $\rho(\lambda)$ as the discontinuity the resolvent $R(\lambda) = (H-\lambda)^{-1}$ of $H$ across the real axis:

$$\rho(\lambda) = \lim_{\epsilon\to 0^+} \frac{1}{i}[R(\lambda+i\epsilon) - R(\lambda-i\epsilon)]. \tag{31}$$

The resolvent in turn can be determined in terms of the Jost solutions [19] of scattering theory. Thus if we have an explicit expression for $V(x)$, we can in principle solve the scattering problem for $H$ and obtain $\rho(\lambda)$.

The static equation of motion implies that $H$ and $M+\epsilon_0$ commute; one way to satisfy this is for $M+\epsilon_0$ and $H$ to be simultaneously diagonal. Generically there will be no degeneracies and this will the only solution. Thus we should expect there to be a real valued function $\sigma(\lambda)$ such that

$$M+\epsilon_0 = \int \sigma(\lambda)\rho(\lambda)\frac{d\lambda}{2\pi}. \tag{32}$$

This function must approach $\pm 1$ as $\lambda \to \pm\infty$ to satisfy the asymptotic condition. Its value in a region not contained in the spectrum of $H$ will not matter to $M$, since $\rho(\lambda)$ will vanish there. Any discrete eigenvalue of $H$ will be contained in an interval in which it is the only element of the spectrum. Thus we can choose to extend $\sigma(\lambda)$ to be a constant in a neighborhood of every discrete eigenvalue. This choice will turn out to be useful later.

We can regard $M(x,y)$ as describing the self–consistent way to fill the energy states of $H$ with fermions.In fact, $\frac{1-\sigma(\lambda)}{2}$ is the filling fraction, the number of states that are occupied as a fraction of the total available ( which is $N$). For large positive $\lambda$, this must go to zero and for large negative $\lambda$ it must go to one; there can be a finite number of states with a fractional filling factor.

Now in our problem, $V(x)$ and $M$ are related by the self–consistency condition,

$$\int_{|p|<\Lambda} \frac{dp}{2\pi}\tilde{M}_{od}(x,p) = \frac{1}{2g^2(\Lambda)}V(x). \tag{33}$$

We have already fixed $g^2(\Lambda)$ from the vacuum solution, so that the r.h.s. is logarithmically divergent. We will now show that for any $V(x)$, the r.h.s. also contains a logarithmically divergent piece; after cancellation of this piece, we will get a convergent equation.

First, note that by a change of variable from $p$ to $\lambda$,

$$\frac{1}{2g^2(\Lambda)} = \int \frac{d\lambda}{2\pi}\frac{\Theta(\lambda^2-m^2)}{k(\lambda)}\Theta(\Lambda^2+m^2-\lambda^2). \tag{34}$$

Here $\Theta$ denotes the step function and

$$k(\lambda) = \surd(\lambda^2 - m^2). \tag{35}$$

Now, the consistency condition can be written as,

$$\int \frac{d\lambda}{2\pi}\left[\Theta(\Lambda^2 - m^2)\frac{\Theta(\lambda^2 - m^2)}{k(\lambda)}V(x) - \sigma(\lambda)\int_{|p|<\Lambda}\frac{dp}{2\pi}\tilde{\rho}_{od}(\lambda; x, p)\right] = 0$$

Now, for large $\Lambda$, the leading contribution of the second term will come from the region of large $|p|$. In the limit of large $p$, the asymptotic behaviour of $\tilde{\rho}(\lambda; , p)$ will be determined by that of the resolvent symbol $\tilde{R}(\lambda; x, p)$. Furthermore, the resolvent symbol for large $|p|$ is,

$$\tilde{R}(\lambda; x, p) \sim [\gamma_5 p + V(x) - \lambda]^{-1} \tag{36}$$

since the WKB approximation applies in this case, for smooth $V(x)$. Thus we find,

$$\tilde{\rho}(\lambda; x, p) \sim \delta(\lambda - \lambda_+)\Pi_+ + \delta(\lambda - \lambda_-)\Pi_- \tag{37}$$

where,

$$\lambda_\pm = \pm\surd(p^2 + |v|^2) \tag{38}$$

and

$$\Pi_\pm = \frac{|v|^2}{|v|^2 + (\lambda_\pm - p)^2}\begin{pmatrix} 1 & \frac{\lambda_\pm - p}{v^*} \\ \frac{\lambda_\pm - p}{v} & \frac{(\lambda_\pm - p)^2}{|v|^2} \end{pmatrix}. \tag{39}$$

In the above, we have assumed without loss of generality that

$$V(x) = \begin{pmatrix} 0 & v(x) \\ v^*(x) & 0 \end{pmatrix} \tag{40}$$

and we have sometimes omitted the $x, p$ dependence of $v, \lambda_\pm, \Pi_p m$ etc. for simplicity. With this asymptotic expression, it can be verified that the divergent terms in the consistency condition cancel out; so we can take the limit as $\Lambda \to \infty$ to get

$$\int \frac{d\lambda}{2\pi}\left[\sigma(\lambda)\rho_{od}(x, x; \lambda) - \Theta(\lambda^2 - m^2)\frac{V(x)}{k(\lambda)}\right] = 0. \tag{41}$$

We have used here, $\int \frac{dp}{2\pi}\tilde{\rho}_{od}(x, p) = \rho_{od}(x, x)$. Although each term separately would be divergent, the quantity in the square brackets has a finite integral over $\lambda$. (This can also be verified using the explicit forms given below.)

It remains now to solve the above nonlinear integral equation for $V(x)$ and $\sigma(\lambda)$. We will make a guess and see if it in fact satisfies the equation. There must be a systematic

method using the inverse scattering theory, [19] but we will not attempt to develop this here.

We propose the ansatz,

$$V(x) = m\gamma_0 + \frac{m}{\eta(x)}(\gamma_\theta - \gamma_0) \tag{42}$$

where,

$$\eta(x) = 1 + e^{-\nu x}, \quad \nu = 2m\sin\frac{\theta}{2} \tag{43}$$

and

$$\gamma_\theta = Q^{-1}(\theta)\gamma_0 Q(\theta), \quad Q(\theta) = e^{i\gamma_5\theta/2}. \tag{44}$$

As $x \to -\infty$, it tends to the vacuum solution, $V(x) \to m\gamma_0$. As $x \to \infty$, it tends to another vacuum solution, differing by a chiral rotation :$V(x) \to e^{\frac{-iga_5\theta}{2}} m\gamma_0 e^{\frac{i\gamma_5\theta}{2}}$. Thus this ansatz describes a sort of topological soliton of our theory.

This ansatz for $V(x)$ is the well-known [19] reflectionless potential of the Dirac operator, which is known to be a soliton of the nonlinear Schrodinger equation. We will use these potentials to produce solitons of our bi–local field theory. There is no direct relationship between the nonlinear Schrodinger equation and our bi-local theory: indeed our theory is relativistically invariant while the nonlinear Schrodinger equation is invariant under Galilean transformations. Yet, the static solution of both systems involve the same reflectionless potential!.

The point spectrum of $H = -i\gamma_5\frac{\partial}{\partial x} + V(x)$ consists of one normalizable eigenstate ('bound state'),

$$\psi_B(x) = \sqrt(\frac{\nu}{2})\frac{e^{\frac{-\nu x}{2}}}{\eta(x)}\begin{pmatrix} 1 \\ ie^{i\frac{\theta}{2}} \end{pmatrix} \tag{45}$$

with eigenvalue $\lambda_B = m\cos\frac{\theta}{2}$. Note that $-m \leq \lambda_B \leq m$, so that the bound state is in the 'gap', the set of values that are forbidden as eigenvalues of the free Dirac operator. The continuum eigenfunctions ( scattering solutions) can be written in terms of the Jost functions, [19] and we can obtain the answer for $\rho(\lambda)$ explicitly. We omit the computations and just display the answer *:

$$\begin{aligned}\rho(x,y;\lambda) = {} & 2\pi\delta(\lambda-\lambda_B)\psi_B(x)\psi_B^\dagger(y) + \\ & \Theta(\lambda^2-m^2)\frac{\text{sgn }(\lambda)(\lambda+k(\lambda))}{2k}\begin{pmatrix} \alpha(x)\alpha^*(y)+\beta^*(x)\beta(y) & \beta^*(x)\alpha(y)+\alpha(x)\beta^*(y) \\ \beta(x)\alpha^*(y)+\alpha^*(x)\beta(y) & \alpha^*(x)\alpha(y)+\beta(x)\beta^*(y) \end{pmatrix}.\end{aligned}$$

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* from now on we use a basis in which $\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \gamma_0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

Here,

$$\alpha(x) = e^{ik(\lambda)x}[1 + \frac{e^{i\frac{\phi-\theta}{2}} - 1}{\eta(x)}], \quad \beta(x) = \frac{im}{k+\lambda} e^{ik(\lambda)x}[1 + \frac{e^{i\frac{\phi+\theta}{2}} - 1}{\eta(x)}]. \tag{46}$$

Moreover, $\phi$ is a sort of scattering phase shift and is determined by the transcendental equation

$$\frac{m}{k+\lambda} \sin\frac{\phi+\theta}{4} = \sin\frac{\phi-\theta}{4}. \tag{47}$$

We only need the special case $\rho_{od}(x,x)$ to verify the equation of motion. We put in as part of the ansatz, $\sigma(\lambda) =$ sgn $(\lambda)$ in the continuum, $|\lambda| \geq m$. After some calculations the consistency condition becomes,

$$\sigma(\lambda_B)\frac{\nu}{2}\frac{e^{-\nu x}}{\eta^2(x)} i e^{i\frac{\theta}{2}} =$$

$$\int \frac{d\lambda}{2\pi}\Theta(\lambda^2 - m^2)\frac{im}{k}[(1 + \frac{e^{i\theta} - 1}{\eta(x)}) - (1 + \frac{e^{i\frac{\theta-\phi}{2}} - 1}{\eta(x)})(1 + \frac{e^{i\frac{\theta+\phi}{2}} - 1}{\eta(x)})]$$

Using $\frac{e^{-\nu x}}{\eta^2(x)} = \frac{1}{\eta(x)} - \frac{1}{\eta^2(x)}$ we see that the terms are proportional to either $\frac{1}{\eta(x)}$ or $\frac{1}{\eta^2(x)}$ ( Terms independent of $\eta(x)$ cancel out.) The terms proportional to $\frac{1}{\eta(x)}$ give, after some simplifications,

$$\sigma(\lambda_B) m \sin\frac{\theta}{2} = -\int \frac{d\lambda}{2\pi}\frac{m}{k}\Theta(\lambda^2 - m^2) 4 \sin\frac{\phi+\theta}{4}\sin\frac{\theta-\phi}{4}. \tag{48}$$

The terms proportional to $\frac{1}{\eta^2(x)}$ happen to give the same equation. With $\sigma(\lambda_B)$ given by the above equation we have a solution to the static equations of motion.

The expression for $\sigma(\lambda_B)$ can be simplified further. If we hold $\theta$ fixed, the scattering angle $\phi$ can be thought of as a function of $\lambda$: either using the previous transcendental equation, or the alternative forms,

$$\cot\frac{\phi-\theta}{4} = \cot\frac{\theta}{2} + \frac{k+\lambda}{m\sin\frac{\theta}{4}}, \quad \tan\frac{\phi}{4} = \frac{k+\lambda+m}{k+\lambda-m}\tan\frac{\theta}{4}. \tag{49}$$

If we change the variable in the integral from $\lambda$ to $\phi$, using the differential of the first one of the above equations

$$-\frac{d\phi}{4\sin^2\frac{\phi-\theta}{4}} = [1 + \frac{\lambda}{k}]\frac{d\lambda}{m\sin\frac{\theta}{2}},$$

we will get

$$\sigma(\lambda_B) m \sin\frac{\theta}{2} = -4\int \frac{d\lambda}{2\pi}[1 + \frac{\lambda}{k}]\Theta(\lambda^2 - m^2)\sin^2\frac{\theta-\phi}{4}$$

$$= \frac{1}{2\pi} m \frac{\sin\theta}{2}\Big[\int_{\lambda=-\infty}^{\lambda=-m} d\phi + \int_{\lambda=m}^{\lambda=\infty} d\phi\Big].$$

Since $\phi = -\theta, 0, 2\pi, \theta$ respectively at $\lambda = -\infty, -m, m, \infty$, we have

$$\sigma(\lambda_B) = \frac{\theta - \pi}{\pi}. \tag{50}$$

To summarize, a static solution of the classical equations of motion is determined by two real functions $\sigma(\lambda)$ and $V(x)$ satisfying the self–consistency equation

$$\int \frac{d\lambda}{2\pi}\left[\sigma(\lambda)\rho_{od}(x,x;\lambda) - \Theta(\lambda^2 - m^2)\frac{V(x)}{k(\lambda)}\right] = 0 \tag{51}$$

where $\rho(\lambda)$ is the spectral density of $H = -i\gamma_5\frac{\partial}{\partial x} + V(x)$. This integral is convergent if $\sigma(\lambda) \sim$ sgn $(\lambda)$ for large $|\lambda|$; no cut–of is necessary. The classical static solution for $M$ is then

$$M = -\epsilon_0 + \int \sigma(\lambda)\rho(\lambda)\frac{d\lambda}{2\pi}. \tag{52}$$

We proposed the ansatz,

$$V(x) = m\gamma_0 + \frac{m}{\eta(x)}(\gamma_\theta - \gamma_0) \tag{53}$$

where,

$$\eta(x) = 1 + e^{-\nu x}, \quad \nu = 2m\sin\frac{\theta}{2}, \quad \gamma_\theta = Q^{-1}(\theta)\gamma_0 Q(\theta), \quad Q(\theta) = e^{i\gamma_5\theta/2}. \tag{54}$$

Then, we showed that

$$\sigma(\lambda) = \text{ sgn } (\lambda) \text{ for } |\lambda| \geq m, \quad \sigma(m\cos\frac{\theta}{2}) = \frac{\theta - \pi}{\pi} \tag{55}$$

satisfies the self–consistency relation. (The value of $\sigma(\lambda)$ can be arbitrarily chosen for other values of $\lambda$, since $\rho(\lambda)$ vanishes there.) Thus we have a one–parameter family of static solutions of the classical theory. The parameter $\theta$ is a topological number that determines the behaviour of the solution as the center of mass variable goes to infinity. When $\theta = 0$, we recover the vacuum solution, $V(x) = m\gamma_0$.

Although we have an explicit expression for $\rho(\lambda; x, y)$, it appears cumbersome to evaluate the integral over $\lambda$ for $M(x, y)$. What we mean by an exact solution is the above integral representation for $M$.

Next we find the energy of this configuration. The integral for energy appears to be too hard to be calculated directly. We will instead use an indirect method. The variation of

$E(M)$ under an infinitesimal change is $\mathrm{tr}H(M)\delta M$. At a static solution, $E(M)$ is invariant under all infinitesimal variations of $M$ satisfying the constraint but that do not change its boundary conditions. However a change of $\theta$ in the above ansatz will change the behaviour of $M$ at infinity, and $E(M)$ need not be stationary with respect to this variation. ( $\theta$ is a topological quantum number of the soliton.) We will get a simple expression for the derivative of energy with respect to $\theta$.

Let us calculate therefore

$$\begin{aligned}\frac{dE(\theta)}{d\theta} &= \mathrm{tr}H(M)\frac{d}{d\theta}(M+\epsilon_0)\\ &= \mathrm{tr}H(M)\int\frac{d\lambda}{2\pi}[\frac{d\sigma(\lambda)}{d\theta}\rho(\lambda)+\sigma(\lambda)\frac{d\rho(\lambda)}{d\theta}]\end{aligned}$$

Now we use the formula,

$$\int\frac{d\lambda}{2\pi}\sigma(\lambda)\frac{d\rho(\lambda)}{d\theta} = \int\frac{d\lambda d\lambda'}{(2\pi)^2}\frac{\sigma(\lambda)-\sigma(\lambda')}{\lambda-\lambda'}\rho(\lambda)\frac{dH}{d\theta}\rho(\lambda') \tag{56}$$

which follows from first order perturbation theory. The second term in the expression for $\frac{dE}{d\theta}$ becomes,

$$I = \int\frac{d\lambda d\lambda'}{(2\pi)^2}\frac{\sigma(\lambda)-\sigma(\lambda')}{\lambda-\lambda'}\mathrm{tr}\rho(\lambda')H\rho(\lambda)\frac{dH}{d\theta}. \tag{57}$$

The trace will be non–zero only if $\lambda=\lambda'$, since $H$ commutes with $\rho(\lambda')$. Then we will have

$$I = \int\frac{d\lambda}{2\pi}\frac{\partial\sigma(\lambda)}{\partial\lambda}\mathrm{tr}H\rho(\lambda)\frac{dH}{d\theta}. \tag{58}$$

Now $\sigma(\lambda)$ is a constant on the continuum, so there is no contribution from it to this integral. As for the bound state, recall that ( for $\theta\neq 0,2\pi$) it is always separated by a gap from the continuum. We can continue $\sigma(\lambda)$ into this gap arbitrarily since $\rho(\lambda)$ is zero there; we could for example choose $\sigma(\lambda)$ to be constant in some interval containing $\lambda_B$. Thus we see that $I=0$. This can probably also be seen by more direct but tedious calculations. We have,

$$\frac{dE(\theta)}{d\theta} = \mathrm{tr}H(M)\int\frac{d\lambda}{2\pi}\frac{d\sigma(\lambda)}{d\theta}\rho(\lambda) = \frac{1}{\pi}\lambda_B = \frac{1}{\pi}m\cos\frac{\theta}{2}. \tag{59}$$

Thus we find ( recall that the configuration with $\theta=0$ is the vacuum which has zero energy)

$$E(\theta) = \frac{2}{\pi}m\sin\frac{\theta}{2}. \tag{60}$$

This agrees with the large $N$ limit of the spectrum of the Thirring model as obtained by the Bethe ansatz method, if we identify $\mu = \frac{2}{\pi}m$.

By similar arguments we can also obtain the derivative of the energy density,

$$\frac{\partial E(x,\theta)}{\partial\theta} = \frac{2}{\pi}m\cos\frac{\theta}{2}\psi_B^\dagger(x)\psi_B(x) \tag{61}$$

which leads to

$$E(x,\theta) = \frac{1}{\pi}[\frac{1}{x}\sin\frac{\theta}{2}\tanh(mx\sin\frac{\theta}{2}) - \frac{1}{mx^2}\log(\cosh(mx\sin\frac{\theta}{2}))] \tag{62}$$

This energy density is peaked around the origin and vanishes exponentially at infinity. This is precisely what we expect for a soliton. It should also be clear that the soliton quantum number $\theta$ corresponds to the (large $N$ limit of) the discrete conserved quantum number of the fermionic theory.( However, the 'baryon number' $B = -\mathrm{tr}M$ is zero for all these configurations. This can be verified by using the regularization methods of [20]. Thus the situation is the opposite of that in 2dqcd; it is the abelian part of the symmetry that is 'confined'.)

There are many additional properties of these solitons that can be studied. For example it would be interesting to obtain the time dependent solutions that represent the classical scattering of several of them. It would also be interesting to generalize the inverse scattering methods of Ref. [19] to infinite component classical field theories such as ours. Furthermore, it should be possible to generalize our results to Thirring models with several flavors.

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